Lecture Notes on Theory of Finance I

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I. Risk, Return, and Risk Aversion

1. Expected Return, Variance and Covariance

Example. Suppose that you have two risky assets: the stock X and gold G and face the following three scenarios:

	Recession	Normal	Boom
Probability	0.20	0.50	0.30
Return of Stock X, r_X	-14%	12%	38%
Return of Gold, r_G	45%	10%	-25%

- **Expected return** is average or mean return you can expect to earn on your investment:

  $$E(r) = \sum_s p(s)r(s),$$

  where $p(s)$ is the probability of the scenario s, $r(s)$ is the return under scenario s.

  $$E(r_X) = 0.20 \times (-14\%) + 0.5 \times (12\%) + 0.30 \times (38\%) = 14.6\%$$

  $$E(r_G) = 0.20 \times (45\%) + 0.5 \times (10\%) + 0.30 \times (-25\%) = 6.5\%$$

- **Variance** measures the average of squared deviation from the average return:

  $$\sigma^2 = \sum_s p(s)[r(s) - E(r)]^2$$

  $$\sigma_X^2 = 0.20 \times (-14 - 14.6)^2 + 0.5 \times (12 - 14.6)^2 + 0.30 \times (38 - 14.6)^2 = 331.24(\%)^2$$

- **Standard deviation** is a square root of variance:

  $$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{331.24} = 18.2\%.$$

  Asset risk is often measured by the standard deviation of the asset's returns.

Practice Problem

What are the variance and standard deviation of the return on gold?

$$\sigma_G^2 =$$

$$\sigma_G =$$

- We are also interested in the co-movement of returns on different assets. The co–movement is described by **covariance**

$$Cov(r_X, r_G) \equiv \sigma_{XG} = \sum_s p(s)(r_X - E(r_X))(r_G - E(r_G))$$

- Covariance measures how much returns move together. In the above example

$$\begin{aligned} Cov(r_X, r_G) &= 0.20 \times (-14 - 14.6) \times (45 - 6.5) + 0.5 \times (12 - 14.6) \times (10 - 6.5) \\ &+ 0.30 \times (38 - 14.6) \times (-25 - 6.5) = -445.9(\%)^2 \end{aligned}$$

- Question: What does the negative covariance tells us about the movement of returns of stock X and gold relative to each other?

- Questions: In what units covariance could be measured? What is the relation between covariances measured in different units?

- Note that covariance of returns of stock X and Gold is the same as covariance of returns of Gold and stock X.

- The magnitude of the covariance is difficult to interpret. Therefore, it is more common to express co-variability using the *correlation coefficient*, ρ , which is covariance divided by the product of the two standard deviations:

$$\rho_{XG} = \frac{Cov(r_X, r_G)}{\sigma_X \sigma_G}$$

In our example:

$$\rho_{XG} = \frac{-445.9}{18.2 \times 24.5} = -1$$

- The correlation coefficient is always between -1 and +1.
- If correlation coefficient is equal to +1.0, the two variables are perfectly positively correlated
- if correlation coefficient is equal to -1.0, the two variables are perfectly negatively correlated
- A correlation coefficient of 0 means that there is no linear relationship between the variables.
- Important formulas on variance and covariance:

$$Var(w_1 r_1 + w_2 r_2) = w_1^2 Var(r_1) + w_2^2 Var(r_2) + 2 w_1 w_2 \, Cov(r_1, r_2)$$

$$Cov(w_1 r_1 + w_2 r_2, r_3) = w_1 \, Cov(r_1, r_3) + w_2 \, Cov(r_2, r_3)$$

$$Cov(r_1, r_1) = \qquad\qquad ,$$

where r_j, $(j = 1, 2, 3)$ is a random return, while w_1 and w_2 are some constants (e.g., weights of stock 1 and 2 in portfolio)

- In practice we use past observations to calculate the expected returns, standard deviations and covariances of stocks
- Assuming that all past events were equally likely, the formulas can be rewritten as:
  - An estimate of expected (average) return

$$\bar{r} = \frac{1}{n} \sum_{t=1}^{n} r_t$$

– An estimate of variance

$$\bar{\sigma}^2 = \frac{n}{n-1}\sum_{t=1}^{n}\frac{1}{n}(r_t - \bar{r})^2$$

(note: $n/(n-1)$ corrects for statistical bias)

– An estimate of covariance between returns of stock X and Y

$$\bar{\sigma}_{XY} = \frac{1}{n-1}\sum_{t=1}^{n}(r_{Xt} - \bar{r}_X)(r_{Yt} - \bar{r}_Y)$$

– Formula for correlation coefficient does not change: $\bar{\rho}_{XY} = \frac{\bar{C}ov(r_X, r_Y)}{\bar{\sigma}_X \bar{\sigma}_Y}$

- Generally, an investor updates the estimates of expected returns from the past with his/her beliefs on future returns

- If there are $m > 1$ stocks then covariances of all possible pairs of stock returns can be described by a variance–covariance matrix, which has variances as its diagonal elements and covariances as its off–diagonal elements. For example if $m = 3$ then a variance–covariance matrix is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{pmatrix}$$

- Covariance matrix is symmetric since $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$ and so on.

- If portfolio has n stocks then its variance–covariance matrix requires finding $n(n+1)/2$ terms

Practice Problem

Calculate estimates of expected returns, variances and covariance of two stocks A and B by using the following returns in the past

Year	Stock A	Stock B
2009	0.25	-0.32
2010	-0.05	0.325
2011	0.31	0.175

$\bar{r}_A =$

$\bar{r}_B =$

$\bar{\sigma}_A^2 =$

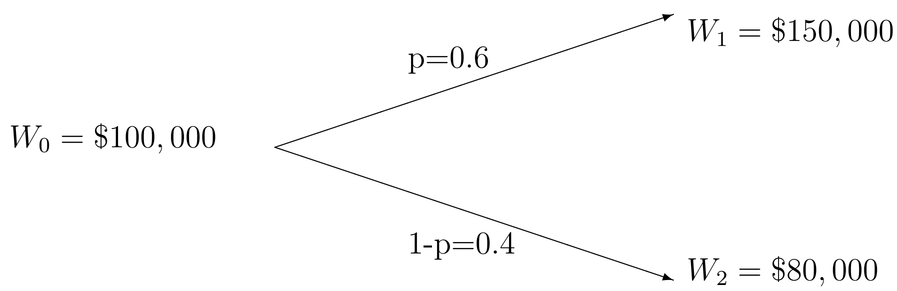
$\bar{\sigma}_{AB} =$

The above calculations can be easily done by using a software package (for example, Excel).

2. Risk and Risk Aversion

- Presence of risk means that more than one outcome is possible. Let us consider an example where there are only two outcomes.

  Suppose you are offered an investment that requires an initial investment of $W_0 = \$100,000$ (which is exactly equal to your current wealth) and which offers the following payoff in one year: with 60% probability you will have $\$150,000$, with 40% probability you will have $\$80,000$. See the diagram below.



- How can you evaluate such an offer?

  First, we summarize it using descriptive statistics:

$$\begin{aligned} E(r) &= p \times r_1 + (1-p) \times r_2 = p \times (W_1 - W_0)/W_0 + (1-p) \times (W_2 - W_0)/W_0 \\ &= (E(W) - W_0)/W_0 = (\$122,000 - \$100,000)/\$100,000 = 22\%, \end{aligned}$$

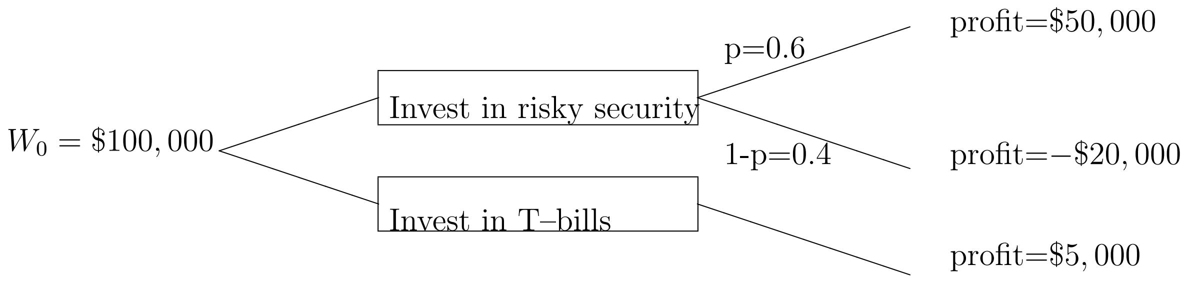
$$\begin{aligned} \sigma_r^2 &= p \times [r_1 - E(r)]^2 + (1-p) \times [r_2 - E(r)]^2 \\ &= 0.6 \times [(150,000 - 100,000)/100,000 - 0.22]^2 \\ &+ 0.4 \times [(80,000 - 100,000)/100,000 - 0.22]^2 = 0.1176 \\ \Rightarrow \sigma_r &= 34\% \end{aligned}$$

- It follows that the standard deviation is larger than the expected return. Therefore the investment is quite risky. Wether the risk is justifiable depends on the alternative portfolios.

- An alternative portfolio could be comprised of only risk–free security

  - Risk–free security has zero variance
  - In practice a risk–free security is approximated by a Treasury bill, which is a bond issued by Government
  - Treasury bills have negligible risk of default but they are still risky (the standard deviation of their annual return is of the order of 3%), for example, due to business cycle risk
  - **Henceforth, we assume the Treasury bills are risk free with a return r_f**
  - We also assume that T–bills can be issued and purchased by any member of the market. So anybody can borrow and lend at rate r_f.

- Let us suppose that Treasury bills have return of 5%. Then our decision tree is shown below



- Because the expected return from the risky project was 22%, **risk premium** or expected marginal return of the risky security over investing in risk free T-bills, is $22\% - 5\% = 17\%$

- A risky investment that has a zero–risk premium is called a *fair game*

- A risky investment that has a negative premium is called a *gamble*

- Question: is the above risk premium sufficient to induce you to invest in the risky project?

- To answer this question we need to introduce utility function

Utility Function

- Assume that investors can assign a welfare or utility score to competing investment portfolios based on their expected return and risk.

- This utility score can then be used in ranking portfolios

- Investors assign higher utility values to portfolios with more attractive risk-return characteristics

- One particular form of utility function that has been widely used in finance is

$$U = E(r) - \frac{1}{2}A\sigma_r^2$$

where U is the utility value and A is **coefficient of risk aversion**, r is a rate of return and σ_r is the standard variation of the rate of return

- The utility function above is also called mean–variance utility
- Investors will choose a portfolio that provides the highest utility level.
- The sign of coefficient of risk aversion defines the type of an investor:
  - If $A > 0$ then an investor is risk–averse.
    * An investor is called risk–averse if he/she rejects investments that are fair games or worse.
    * Risk–averse investor would like to have higher expected return and lower standard deviation. If risk of an investment increases then an investor demands higher expected return
    * Most of investors are risk–averse.
  - If $A = 0$ then an investor is risk–neutral.
    * An investor is called risk–neutral if he/she judges risky investments solely by their expected rate of return.
    * Risk–neutral investor does not care for risk of investment.
  - If $A < 0$ then an investor is risk–loving.
    * An investor is called risk–loving if he/she is willing to engage in gambles.
    * Risk–loving investor likes to have higher both expected return and standard deviation. If expected return of an investment falls then an investor demands higher risk
- *Assumption*: **From now on we assume that all investors are risk–averse**
- A is measured in decimals

- Question: Which units we use for $E(r)$ and σ_r to find U?

- Let's go back to the previous example. We found that $E(r) = 22\%$, $\sigma_r^2 = 0.1176$
  - Assume that your coefficient of risk aversion, A, is 4. Then

$$U = E(r) - \frac{1}{2}A\sigma_r^2 = 0.22 - \frac{1}{2} \times 4 \times 0.1176 = -0.0152$$

  - Question: will you invest in the risky project?

    Answer: , because U(risk-free) U(risky)

Practice Problem

Will you invest in the risky project if your A is 2? Give an economic intuition.

- Besides the coefficients of risk aversions, investors also differ in their beliefs. For example, Michael expects that GM stock will grow at 12% with $\sigma_r = 10\%$, while Elizabeth expects these numbers to be 8% and 15%, respectively.

- For the most part of this course we assume that beliefs of investors on future returns are identical

Indifference Curve

- Indifference curve is a line that connects all portfolios (in the expected return–standard deviation plane) that provide the same level of utility

- In other words, all portfolios that lie on the same indifference curve are equally attractive to an investor

- Indifference curve is a graphical way of describing preferences of an investor

- Let's find a typical indifference curve on the graph with axes measuring the expected value and standard deviation of portfolio returns. Assume that the level of expected utility is U_0. Then, $U_0 = E(r) - \frac{1}{2}A\sigma_r^2$ implies

$$E(r) = U_0 + \frac{1}{2}A\,\sigma_r^2$$

  which is an equation of parabola on $(E(r), \sigma_r)$–plane.

- The last equation gives us all possible combinations of $E(r)$ and σ_r (that is, portfolios) that provide the same level of utility U_0.

- Let us plot the indifference curve assuming that $U_0 = 0.05$ and $A = 4$. From the equation $E(r) = 0.05 + 2\sigma_r^2$ we find a few points lying on the curve:

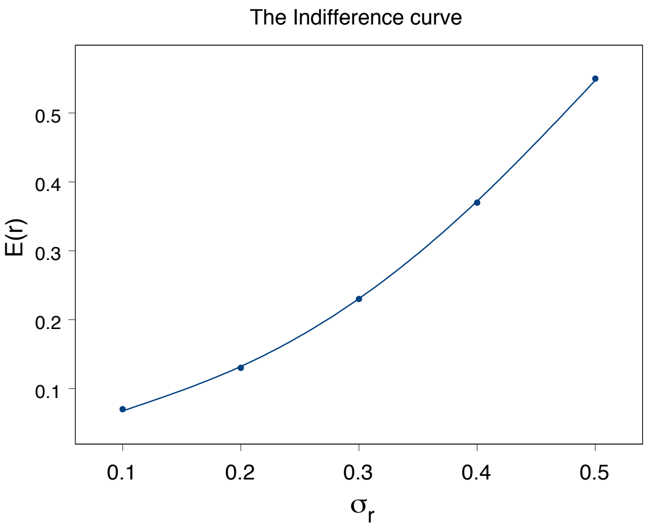
σ_r	$E(r)$
0.1	0.07
0.2	0.13
0.3	0.23
0.4	0.37
0.5	0.55

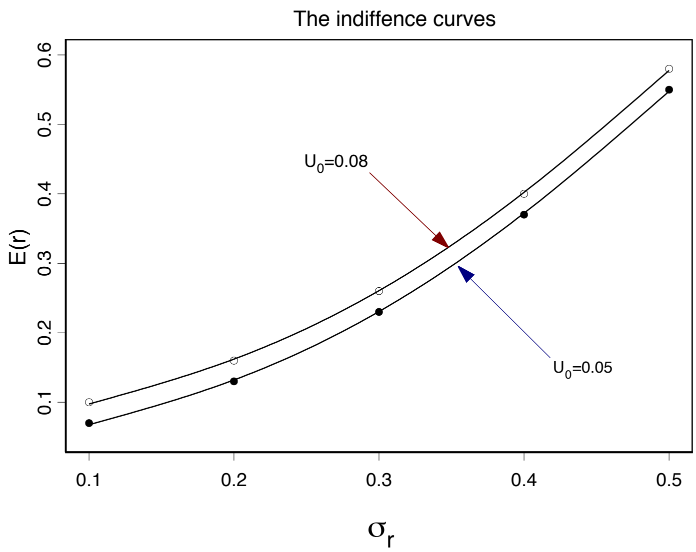
  These points are shown on the graph below

- Suppose that we have to replicate the above table for $U_0 = 0.08$.

  – Question: where would new indifference curve lie? Above or below the one for $U_0 = 0.05$?

  – Answer:





Practice Problem

Fill in the table below if the coefficient of risk aversion, A, is equal to 2 and the level of expected utility function, U_0 is fixed.

σ_r	$E(r)$	U_0
0.1		
0.2	0.1	
0.3		

– Assume that A is changed to 3 while U_0 is unchanged, where would an indifference curve for such an individual lie? Above or below the one for A = 2? Why?

– Answer:

- We conclude that the higher the risk aversion coefficient A the steeper is the indifference curve
- Questions: Can indifference curves for the same investor intersect?
  Can indifference curves for different investors intersect?

Portfolio Risk

- Let's consider a portfolio that includes two assets A and B
- Expected return on a portfolio is

$$E(r_p) = w_A E(r_A) + w_B E(r_B),$$

where w_A = fraction of portfolio invested in asset A and w_B = fraction of portfolio invested in asset B

- Portfolio variance is

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \rho_{AB},$$

where ρ_{AB} is a correlation coefficient between assets A and B

- In the special case when asset A is risky but asset B is riskless portfolio variance becomes

$$\sigma_p^2 = w_A^2 \sigma_A^2$$

Practice Problem

Suppose we are given the following information on stocks A and B:

	$E(r)$	σ
Stock A	15%	20%
Stock B	10%	30%

Let's examine some portfolios that combine stocks A and B:

- Assume that $\rho_{AB} = 0$, what would be the expected return and standard deviation on a portfolio with 80% in stock A and 20% in stock B?

  $E(r_p) =$

  $\sigma_p =$

  Question: Is σ_p larger or smaller than σ_A and σ_B?

  Answer:

- Assume that $\rho_{AB} = -1$, what would be the standard deviation on a portfolio with 80% in stock A and 20% in stock B?

  $\sigma_p =$

  Question: Is σ_p larger or smaller than σ_A and σ_B?

  Answer:

- Risk–averse investors often seek the ways of reducing the risk. It could be reduced by **diversification** and **hedging**

- Diversification is a strategy of investing in wide variety of assets so that the exposure to the risk of any particular security is limited
- Hedging is investing in an asset with a payoff pattern that offsets our exposure to a particular source of risk. For example, an insurance contract
- Some textbooks abuse the difference between the two terms and call both diversification

Additional Practice Problems

Use the following to answer questions 1 and 2:

You have been given this probability distribution for the holding period return for XYZ stock:

State of the Economy	Probability	HPR (%)
Boom	0.30	18
Normal growth	0.50	12
Recession	0.20	-5

1. What is the expected holding period return for XYZ stock?

A) 11.67%

B) 8.33%

C) 10.4%

D) 12.4%

E) 7.88%

2. What is the expected standard deviation for XYZ stock?

A) 2.07%

B) 9.96%

C) 7.04%

D) 1.44%

E) 8.13%

3. The standard deviation of return on investment A is 40% while the standard deviation of return on investment B is 20%. What is the correlation coefficient between returns on A and B if the covariance of returns on A and B is 0.050?

A) 1

B) 0.5

C) 0.625

D) 0

E) 0.75

4. Consider a portfolio with 70% in stock A and 30% in stock B.

Stock A: $E(r_A) = 0.15, \quad \sigma_A = 0.20$

Stock B: $E(r_B) = 0.10, \quad \sigma_B = 0.05$

$\rho_{AB} = 0.05$

Find the expected return and the standard deviation of the portfolio returns

5. In a return-standard deviation space, which of the following statements is (are) true for risk-averse investors? (The vertical and horizontal lines are referred to as the expected return-axis and the standard deviation-axis, respectively.)

I) An investor's own indifference curves might intersect.

II) Indifference curves have negative slopes.

III) In a set of indifference curves, the highest offers the greatest utility.

IV) Indifference curves of two investors might intersect.

A) I and II only

B) II and III only

C) I and IV only

D) III and IV only

E) none of the above

6. Suppose you currently hold a portfolio that yields 15% return and has a standard deviation of 20%. What should the minimum rate of return on a portfolio with standard deviation of 25% be to make you indifferent between the two portfolios? Assume your coefficient of risk aversion is 3.

7. A portfolio is comprised of two stocks, A and B. Stock A has a standard deviation of return of 20% while stock B has a standard deviation of return of 5%. Stock A comprises 70% of the portfolio while stock B comprises 30% of the portfolio. If the variance of the return on the portfolio is 0.0023, what is the correlation coefficient between returns on A and B?